Flow resistance of inertial debris flows

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Abstract

This work deals with the evaluation of the most suitable expression for the motion resistance of a debris flow. In particular, we focus on inertial debris flows, i.e., granular-fluid mixtures in which the particle inertia dominates both the fluid viscous force and turbulence; we provide, through an order of magnitude analysis, the criterion to be satisfied for a debris flow to be considered inertial and we show that most of real scale debris flows match this description. We then use the analytical relation between flow depth, depth-averaged velocity and tangent of the angle of inclination of the free surface recently obtained by Berzi and Jenkins in steady, uniform flow conditions to approximate the flow resistance in depth-averaged mathematical models of debris flows. We test that resistance formula against experimental results on the longitudinal profile of steady, fully saturated waves of water and gravel over both rigid and erodible beds, and against field measurements of real events. The notable agreement, especially in comparison with the results obtained using other resistance formulas for debris flows proposed in the literature, assesses the validity of the theory.

Introduction

Two-phase, depth-averaged mathematical models seem to be a useful tool to predict the propagation of a debris flow (Iverson 1997; Pitman and Le 2005), here defined as a dense (i.e., high concentrated) mixture of water and solid particles, driven down a slope by gravity. In this context, two different expressions for the depth-averaged resistances of the fluid and the particles should be provided. Also, the depth-averaged mathematical models should allow for the fluid and particle depths being different, as experimentally shown by Armanini et al. (2005) and Iverson et al. (2010). Different physical mechanisms contribute to the development of shear stresses, and therefore flow resistance, in particle-fluid mixtures: the fluid viscous force, the fluid turbulence, the inter-particle collisions and frictional contacts. The latter two are dominant in what we call 'inertial debris flows', which are the focus of the present paper. This is a wider definition with respect to the inertial regime described by Bagnold (1954), where only the inter-particle collisions were taken into account. On the other hand, we call 'mudflows' the debris flows dominated by the fluid viscous force. The fluid turbulence is negligible in the case of debris flows, because of the high particle concentration.

Recently, Berzi and Jenkins (2008a,b, 2009) have developed a two-phase theory to analytically describe the behavior of debris flows in steady, uniform and non-uniform flow conditions, when the degree of saturation (ratio of fluid to particle depth) is allowed to differ from unity; they successfully compared their analytical results with the experiments performed by Armanini et al. (2005) and Tubino and Lanzoni (1993) on the flows of water and different types of granular material (plastic cylinders, glass spheres and gravel). In particular, Berzi and Jenkins (2009) provided the expressions, further simplified by Berzi et al. (2010), for the resistance formulas of the two phases (fluid and particles), to be used in depth-averaged mathematical models.

Here, we provide a rational criterion, through an order of magnitude analysis, to define the aforementioned inertial debris flows. The order of magnitude analysis provides further justification of the assumptions made by Berzi and Jenkins in developing their theory. Based on the description provided by Iverson (1997), we also show that most of real scale debris flows can actually be considered inertial. For sake of simplicity, we limit the analysis to fully saturated debris flows, i.e., flows for which the particle and fluid depths above the either rigid or erodible bed (Armanini et al. 2005) coincide. The order of magnitude analysis, though, holds in general for nearly saturated debris flows, i.e., flows for which the fluid and particle depths are slightly different. Limiting the analysis to fully saturated debris flows permits to compare the resistance formula obtained from the theory of Berzi and Jenkins with previous single-phase expressions suggested in the literature. Indeed, despite the rather trivial consideration that the expression of the flow resistance is crucial in mathematically modeling debris flows, a relatively small effort has been devoted to actually evaluate the reliability of the available resistance formulas. The few works on the topic (Hungr 1995; Naef et al. 2006) investigated the influence of the resistance formulas on problems dominated by acceleration and mass exchange phenomena, sometimes making use of real debris flow events as test cases. We claim, on the contrary, that a minimum requirement for a resistance formula is to predict the relation between flow depth, depth-averaged flow velocity and angle of inclination of the free surface observed in a well controlled environment, such as a laboratory, on simple flow configurations, such as steady, uniform, or non-uniform, flows.

The paper is organized as follows. In Section 2, we briefly summarize the governing equations for steady, uniform, and fully saturated debris flows and perform the order of magnitude analysis to identify inertial debris flows and support the theory of Berzi and Jenkins (2008a,b, 2009). Then, in Section 3, we introduce and discuss the most popular resistance formulas so far adopted in mathematical models of debris flows, and we test their capability to predict the longitudinal profile of steady, fully saturated waves of water and gravel over either rigid or erodible beds, experimentally measured by Iverson et al. (2010) and Tubino and Lanzoni (1993). For completeness, we also show comparisons between the predictions of the theory of Berzi and Jenkins and field measurements on inertial debris flows. Finally, we draw some conclusions in Section 4.

Theory

Governing equations

We let ρ denote the fluid mass density, c the particle concentration, g the gravitational acceleration, σ the ratio of particle to fluid density, d the particle diameter, η the fluid viscosity, U the fluid velocity, and u the particle velocity. The particle Reynolds number $\mathrm{R} \equiv \rho d \left(gd\right)^{1/2} / \eta$ is defined in terms of these. In what follows, all quantities are made dimensionless using the particle diameter, the mass density of the particle material, $\rho\sigma$, and the gravitational acceleration. We take $z = h$ to be the free surface, and $z = 0$ to be the position of the bed of inclination θ, parallel to the free surface. The flow configuration is depicted in Fig.1a.

In Table 1, we summarize the momentum balances and the constitutive relations reported by Berzi and Jenkins (2009) for the steady, uniform flow of fluid and particles over a bed, in presence of lateral confinement; s, p, S and D are the particle shear stress, the particle effective pressure (total particle pressure minus pore pressure), the fluid shear stress and the drag exerted by the fluid on the particles, respectively. There, and in what follows, a prime indicates a derivative with respect to z. The additional force exerted on the particles by the vertical sidewalls, separated by a gap of width W, is taken into account on average through their coefficient of sliding friction, μ_w (Berzi and Jenkins 2008a, b).

The expression for the drag is that suggested by Jenkins and Hanes (1998), where $\delta = U - u$, and $3T$ is the mean square of the particle velocity fluctuations, T being the granular temperature.

The adopted particle rheology is a linearization of the phenomenological rheology suggested by the French group G.D.R. MiDi (2004), with the particle stress ratio, s/p, that depends only on the so called inertial number, $I \equiv u' / \left(p / c\right)^{1/2}$. In the linear particle rheology reported on Table 1, χ is a material coefficient of order unity and $\breve{\mu}$ is the tangent of the angle of repose of the dry granular material in absence of lateral confinement (Berzi et al. 2010). The linear particle rheology is supposed to be valid at high particle concentrations (da Cruz et al. 2005). Actually, Jenkins (2007), Jenkins and Berzi (2010) and Berzi and Jenkins (2011) showed that the phenomenological rheology of the G.D.R. MiDi (2004) applies only in a region a few diameters far from the boundaries (i.e., the free surface and the bed, in the present case), and provided the particle rheology in this core region using a more fundamental approach based on kinetic theories of dense granular gases (Jenkins and Savage 1983; Goldhirsch 2003; Jenkins 2006). The particle rheology of Table 1 applies, therefore, to thick debris flows (particle depth greater than, say, ten diameters) characterized by a relatively narrow range of particle stress ratios. Berzi and Jenkins (2009) showed that that narrow range of s/p corresponds, though, to a range of angles of inclination of the bed typical of both laboratory and real scale debris flows; they also showed that the corresponding values of the particle concentration are in the range 0.5 to 0.6, indicating that the flow is dense. This justifies the fact that c is taken constant in the expressions of Table 1.

Finally, a mixing length approach is used to express the turbulent fluid shear stress in Table 1. Berzi and Jenkins (2009) took into account the possibility that either a large-scale (with the mixing length, l, proportional to h) or a small-scale turbulence (with l of the order of the mean distance between the particles) develops in the region where both fluid and particles are present. According to many authors (Bagnold 1954; Derksen 2008), though, the presence of the particles at high concentration suppresses the large-scale turbulence. Thus, we take l to be roughly one tenth of a particle diameter.

Berzi and Jenkins (2009) used, as boundary conditions, the vanishing of the particle and fluid stresses at the free surface. At the bed, instead, boundary conditions for the particle and fluid velocity are required. For the latter, the no-slip condition seems to apply; previous works have instead shown that the particles slip at a rigid bed (Richman 1988; Jenkins 2001), at least in absence of interstitial fluid, with:

$$u_0 \propto \frac{s_0}{p_0}\left(\frac{p_0}{c_0}\right)^{1/2}. \tag{1}$$

In Eq. (1), the sub-index indicates the location z at which a quantity is evaluated. In the case of erodible bed, instead, the no-slip condition applies also to the particles (Berzi and Jenkins 2008a,b).

Order of magnitude analysis

As already mentioned, the linear particle rheology applies to thick and dense granular flows; i.e., flows characterized by h much greater than one and c of order unity; the particle specific mass and the tangent of the angle of inclination of the bed are both of order unity.

Given that the coordinate z is of order h, the particle momentum balance along z (Table 1) shows that p is of order h. The inertial number I is of order 10^{-1}, according to physical and numerical experiments on simple shear flow of dry granular material (G.D.R. MiDi 2004) and recent theory (Berzi et al. 2011), at least when the particle stress ratio is not close to the yielding value $\breve{\mu}$. Hence, from the definition of I, the shear rate u' is of order $10^{-1}h^{1/2}$. This implies that u is of order $10^{-1}h^{3/2}$, also known as the Bagnold scaling (Mitarai and Nakanishi 2005); obviously, also the depth-averaged particle velocity, u_m, is of order $10^{-1}h^{3/2}$.

The granular temperature T scales with p (Jenkins 2007) and therefore is of order h. We now assume that the non-linear part of the drag coefficient in the expression of the drag force (Table 1) is entirely due to the particle velocity fluctuations, i.e., δ is negligible with respect to $T^{1/2}$. This implies that δ is much less than $h^{1/2}$ and, therefore, also much less than u. With this, the fluid and particle velocities would be approximately identical (single-phase approximation), and $u' \approx U'$, as assumed by Berzi and Jenkins (2008a,b, 2009); then, U is of order $10^{-1}h^{3/2}$. The constitutive expression for the fluid turbulent shear stress of Table 1 gives, therefore, that S is of order 10^{-4}h, with l of order 10^{-1}. Thus, S' is of order 10^{-4} and can be neglected with respect to the component of the fluid weight along x (first term on the right hand side of the fluid momentum balance in Table 1), which is of order unity; hence, the fluid momentum balance reduces to a balance between the component of the fluid weight in the flow direction and the drag (with the fluid turbulence having no influence on the flow). The drag must therefore be of order unity. The expression of the drag in Table 1 can then be used to obtain that δ is of order $h^{-1/2}$, which is consistent with our initial guess on δ. If the depth h is not much greater than one, or if the inertial number is much smaller than one, i.e., if the particle stress ratio is close to $\breve{\mu}$, δ cannot be neglected with respect to u and the single-phase approximation no longer holds. The former can be the case for some laboratory debris flows, as shown in Berzi and Jenkins (2008a, b), while the latter is certainly the case at the onset and arrest of debris flows, whose modeling therefore require a full two-phase approach.

Eq. (1) shows also that u_0 is of order $h^{1/2}$ for flows over rigid beds, hence negligible with respect to u_m when h is much greater than one, given that the particle stress ratio, s/p, is of order unity (G.D.R. MiDi 2004; Jop et al. 2005; Mitarai and Nakanishi 2007).

The conditions for the validity of the linear particle rheology (thick flow and high concentration) permit therefore to ignore the difference in velocity between the fluid and the particles, at least far from the onset and the arrest, the particle slip velocity at the rigid bed and the turbulent fluid shear stress, as in Berzi et al. (2010).

Actually, the use of the linear particle rheology (Table 1) has not been justified, yet. That rheology holds for dense and dry granular flows. The interstitial fluid affects the particle interactions at the micro-mechanical level in a significant way if the Stokes number, $\mathrm{St} \equiv \sigma T^{1/2} \mathrm{R}/9$, for the particles is small (Joseph et al. 2001; Courrech du Pont et al. 2003; Berzi 2011). Hence, given that we have shown that $T^{1/2}$ is of order $h^{1/2}$, the influence of the interstitial fluid on the particle interactions can actually be ignored, and the debris flow can be defined inertial, if $\mathrm{St} \approx 10^{-1} \mathrm{R} h^{1/2}$ is much greater than one, i.e., if R is much greater than $10h^{-1/2}$. The typical flow depths of real scale debris flows are of order one meter (Iverson 1997); with this, and using the definition of the Reynolds number, and the values of density and viscosity appropriated for water, the aforementioned condition would imply d much greater than 10^{-3} mm. It is worth mentioning that $d = 0.1$ mm is the silt-sand boundary (Iverson 1997).

According to Iverson (1997), 90% of particles in debris flows is composed of sand, gravel or larger grains; the remaining 10% is composed of finer components, whose main effect is increasing the apparent density and viscosity of the interstitial fluid, without changing though the order of magnitude of ρ and η that we have employed in the present analysis (using the expressions reported by Iverson, ρ and η would be about 1.2 and 1.4 times the corresponding values for clear water, respectively). Hence, most of real scale debris flows are inertial and the theoretical solution to steady, uniform flows reported by Berzi et al. (2010) applies to them.

We now derive, using the above analysis and the expressions of Table 1, the theoretical solution for steady and uniform, fully saturated, inertial debris flows over rigid beds confined between vertical sidewalls. With c approximately constant in the momentum balances of Table 1, the total shear stress of the mixture and the particle pressure read

$$s+S=\frac{\sigma c+1-c}{\sigma}(h-z)\sin\theta-\frac{\mu_w}{W}p(h-z), \qquad (2)$$

and

$$p=\frac{c(\sigma-1)}{\sigma}(h-z)\cos\theta, \qquad (3)$$

respectively. Given that the fluid turbulent shear stress is negligible in Eq. (2), the particle stress ratio results linearly distributed,

$$\frac{s}{p}=\frac{(\sigma-1)c+1}{(\sigma-1)c}\tan\theta-\frac{\mu_w}{W}(h-z). \qquad (4)$$

From the particle rheology of Table 1, also the inertial number is linearly distributed in the flow,

$$I=\frac{1}{\chi}\left[\frac{(\sigma-1)c+1}{(\sigma-1)c}\tan\theta-\frac{\mu_w}{W}(h-z)-\breve{\mu}\right]. \qquad (5)$$

Using the definition of the inertial number and the particle pressure distribution (Eq. 3), we obtain,

$$u'=\frac{1}{\chi}\left[\frac{(\sigma-1)c+1}{(\sigma-1)c}\tan\theta-\frac{\mu_w}{W}(h-z)-\breve{\mu}\right]\left[\frac{(\sigma-1)\cos\theta}{\sigma}\right]^{1/2}(h-z)^{1/2}. \qquad (6)$$

Eq. (6) can easily be integrated to obtain the velocity distribution along z, using the no-slip boundary condition at the rigid bed (in accordance with the order of magnitude analysis),

$$\begin{aligned} u = & \frac{2}{3}\frac{1}{\chi}\left[\frac{(\sigma-1)c+1}{(\sigma-1)c}\tan\theta-\breve{\mu}\right]\left[\frac{(\sigma-1)\cos\theta}{\sigma}\right]^{1/2}\left[h^{3/2}-(h-z)^{3/2}\right] \\ & -\frac{2}{5}\frac{1}{\chi}\frac{\mu_w}{W}\left[\frac{(\sigma-1)\cos\theta}{\sigma}\right]^{1/2}\left[h^{5/2}-(h-z)^{5/2}\right]. \end{aligned} \tag{7}$$

Finally, integrating Eq. (7) between 0 and h allows to obtain the depth-averaged particle velocity, in the case of mild slopes ($\cos\theta \approx 1$),

$$u_m = \frac{2}{5}\frac{1}{\chi}\left[\frac{(\sigma-1)c+1}{(\sigma-1)c}\tan\theta-\breve{\mu}\right]\left[\frac{(\sigma-1)\cos\theta}{\sigma}\right]^{1/2} h^{3/2}-\frac{2}{7}\frac{1}{\chi}\frac{\mu_w}{W}\left[\frac{(\sigma-1)\cos\theta}{\sigma}\right]^{1/2} h^{5/2}. \tag{8}$$

Eq. (8) can be inverted to obtain an expression for the so called friction slope, j, that, in uniform flow conditions, equals $\tan\theta$:

$$j = \frac{(\sigma-1)c}{(\sigma-1)c+1}\breve{\mu}+\frac{5\chi}{2}\frac{\left[\sigma(\sigma-1)\right]^{1/2}c}{(\sigma-1)c+1}\left[\frac{u_m}{h^{3/2}}+\frac{2}{7}\frac{1}{\chi}\frac{(\sigma-1)^{1/2}}{\sigma^{1/2}}\frac{\mu_w}{W}h\right]. \tag{9}$$

It is customary to use the expression of the friction slope obtained in uniform flow conditions to approximate the flow resistance in depth-averaged mathematical models of non-uniform flows (Chow 1959). In this sense, Eq. (9) represents the resistance formula for saturated debris flows over rigid beds in presence of lateral confinement obtained from the theory of Berzi and Jenkins (2008a,b, 2009). The first term on the right hand side of Eq. (9) represents the minimum slope (yield) for having a steady, uniform flow; as expected, it increases as the concentration increases.

We can obtain the resistance formula for saturated debris flows over erodible beds confined between vertical sidewalls by assuming, as in Berzi and Jenkins (2008a), that the particle stress ratio is at its yielding value at the bed. Eq. (4) therefore provides an additional relation to determine the flow depth as a function of the slope,

$$\breve{\mu} = \frac{(\sigma-1)c+1}{(\sigma-1)c}\tan\theta-\frac{\mu_w}{W}h. \tag{10}$$

Using Eq. (10) in Eq. (8), and substituting j for $\tan\theta$ gives

$$j = \frac{(\sigma-1)c}{(\sigma-1)c+1}\breve{\mu} + \frac{35\chi}{4}\frac{\left[\sigma(\sigma-1)\right]^{1/2} c}{(\sigma-1)c+1}\frac{u_m}{h^{3/2}}. \tag{11}$$

It is worth noticing that, in saturated flow conditions, the two-phase theory of Berzi and Jenkins reduces to a single-phase theory (the dimensional analysis has indeed shown that the fluid and the particle velocity are roughly identical, if the flow is thick). This will allow us to compare Eqs. (9) and (11) with the widely used, single-phase, resistance formulas mentioned in the next Section.

Test of resistance formulas

Unfortunately, it is quite difficult to make accurate measurements on granular flows, even in a well controlled environment such as a scientific laboratory. Usually, both the depth and velocity are optically measured through glassy sidewalls, thus influenced by the latter. Also, the determination of the depth is easy in the case of flows over rigid beds, while in the case of flows over erodible beds depends on the location of the bed itself, which is still under debate (Armanini et al. 2005; Jenkins and Berzi 2010; Berzi et al. 2010). We have shown in the previous section that the debris flow is not influenced by the boundaries, if the depth is much greater than, say, ten diameters. This condition is normally achieved in real scale events (Iverson 1997), while all of the available laboratory experiments on uniform debris flows over rigid beds are characterized by depths of roughly ten diameters (Armanini et al. 2005; Hotta and Miyamoto 2008). Experiments characterized by depths of over a hundred diameters are actually reported by Hotta and Miyamoto (2008), but they can be classified as mudflows (R is of order $10h^{-1/2}$), not inertial debris flows. Finally, in the most general case, the depth and velocity of the particles differ from those of the fluid, and they should be measured separately.

To our knowledge, the only experimental campaign with detailed measurements of particle and fluid depths and depth-averaged velocities – calculated from the volume flow rates – and angle of inclination of the free surface was performed by Armanini et al. (2005) on steady, uniform, debris flows over erodible beds; some experiments were also performed by Tubino and Lanzoni (1993), though, in that case, the difference between the fluid and particle depth was not measured. Berzi and Jenkins (2008a,b, 2009) have shown that their two-phase theory was able to predict in a notable way the experimental results of both Armanini et al. (2005) and Tubino and Lanzoni (1993).

As already mentioned, a fair test of the performance of the theory of Berzi and Jenkins against other resistance formulas, based on single-phase approach, should be made using experiments on fully saturated debris flows. Unfortunately, those experiments are rather scarce. An alternative is to analyze steady, fully saturated waves translating along inclines at constant velocity (Fig.1b and 1c). Indeed, the equation describing the shape of a wave moving at constant velocity along a plane is (Pouliquen 1999b; Berzi and Jenkins 2009):

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \tan\theta - j\,. \tag{12}$$

We need an expression for the friction slope, j, – the boundary condition being the vanishing of h at a certain position $x = L$ along the bed – to solve Eq. (12). Apart from the steady, uniform flows, this is therefore the simplest flow configuration that allows to assess the validity of a resistance formula.

A list of the most popular resistance formulas adopted so far in debris flow models is reported on Table 2. The Coulomb resistance formula (Savage and Hutter 1989; Iverson 1997; Pitman and Le 2005) is commonly adopted in Earth Science related works; it is based on the assumption that the granular material slides over an incline as a solid object without internal shearing, with the constant basal friction angle, ϕ, independent on the flow velocity and depth, in contrast with experimental evidence on both dry granular and debris flows (Pouliquen 1999a; Armanini et al. 2005). The Takahashi's (1991) formula has been quite successful in the Hydraulics literature on debris flows; it is based on the pioneering work on inertial granular flows of Bagnold (1954), who correctly described the physical mechanism at the origin of the particle pressure (the particle collisions), but was wrong in assuming a Coulomb-like relation between the particle shear stress and pressure, as clearly proved by recent numerical simulations on simple shear flows (da Cruz et al. 2005). In the expression reported on Table 2, c^* is the concentration at the closest packing, taken to be 0.74 as for mono-dispersed spheres (Torquato 1995), while a is a parameter that takes into account the nature of the bed (rigid or erodible).

For completeness, we have also listed in Table 2 some resistance formulas that, although do not strictly apply to inertial debris flows, have nonetheless been suggested in the literature. As already mentioned, the resistance formulas based on the assumption that the fluid viscous force dominates over the particle inertia may apply to mudflows, not to inertial debris flows. Several rheologies have been proposed (e.g., Newtonian, Bingham, Herschel-Bulkley, Coulomb-viscous; see Naef et al. 2006 for references and a more detailed discussion) to derive those 'viscous' resistance formulas. In Table 2, we report only the resistance formula based on the Newtonian laminar rheology, where $\mathrm{R}^* = \mathrm{R}\left[\left(c^*/c\right)^{1/3} - 1\right]^{3/2}/2.25$ is a modified particle Reynolds number that takes into account the influence of the concentration on the fluid viscosity, as suggested by Bagnold (1954).

On the opposite, there are some resistance formulas that emphasize the 'turbulent' behavior of debris flows (i.e., the Manning-Strickler and Voellmy formulas reported in Table 2; see, once again, Naef et al. 2006). We have already stated in the previous section that the fluid turbulence is likely to be suppressed when the concentration is high; turbulent-like formulas may therefore apply to the flow of fluid-particle mixture at low-moderate concentration, but, once again, not to inertial debris flows. In the expressions of Table 2, n and ξ are the dimensional Manning and Voellmy coefficients, respectively. In the Voellmy formula, a turbulent-like term is added to a yield term; for the latter, we adopt the expression derived by Berzi and Jenkins (2008a,b, 2009) (first term on the right hand side of Eq. 9).

Iverson et al. (2010) reported the aggregated results of 15 experiments, characterized by the same initial conditions, on debris flows of water and a mixture of gravel and sand over rigid, rough beds

(Fig.1b) in a rectangular channel of width, W, equal to 200 cm (200 diameters, given that the mean diameter of the sediments was equal to 1 cm) and constant inclination θ equal to 31°, in terms of wave height as a function of time, t. After an initial acceleration, the velocity of the front of the wave reached a value of about 10 m/s, i.e., $u_m = 32$ in dimensionless units, and remained roughly constant for the most of the length of the channel. There, the wave is therefore approximately steady in a frame of reference moving at constant velocity, with $x = 32t$. Fig.2 shows the comparisons between the average results of the 15 experiments and those obtained by numerically solving Eq. (12), with a fourth-order Runge-Kutta method, using the aforementioned six resistance formulas for j; i.e., Eq. (9) and the five expressions of Table 2. In the latter, we use: $\sigma = 2.65$, appropriated for sand and/or gravel in water; $\breve{\mu} = 0.5$, the tangent of the angle of repose in a channel of infinite width, obtained by Forterre and Pouliquen (2003) for dry sand (assuming that sand and gravel have similar properties); $c = 0.65$, the average value of the concentration of sand near an erodible bed measured by Pugh and Wilson (1999); $\chi = 0.6$, that allows to reproduce the experimental results on debris flows of water and gravel in uniform flow conditions (Berzi et al. 2010); $\tan\phi = 0.8$, as suggested by Iverson et al. (2010); $a = 0.35$, given that the bed is rigid (Takahashi 1991); $n = 0.1$ s/m$^{1/3}$, as suggested by Rickenmann (1999); $\xi = 1120$ m/s^2, as suggested by Buser and Frutiger (1980), analyzing data on snow avalanches. Also, given that the channel width is about 20 times larger than the flow depth, we ignore the additional term due to the presence of sidewalls in Eq. (9). The particle Reynolds number in the experiments of Iverson et al. (2010) is about 3100 (with $\eta = 10^{-3}\,\mathrm{Pa \cdot s}$); given that h is of order ten diameters (Fig.2), R is much greater than $10h^{-1/2}$ and the particle inertia dominates the flow. The roughness of the rigid bed helps to greatly reduce the slip velocity of the particles, so that the conditions for the validity of the theory of Berzi and Jenkins are probably satisfied, despite the fact that the flow is not really thick. The agreement between the experimental and the theoretical wave profile obtained using Eq. (9) is notable in terms of the maximum height reached by the wave; even more notable, if one keeps in mind that the experimental data are characterized by a significant dispersion and that the theory was developed for a mono-dispersed mixture of particles and water. On the other hand the reproduction of the shape of the snout is less satisfactory. There the depth is less than ten diameters, so that the influence of the bottom boundary cannot be neglected: the rough bed acts as a source of energy to the flow (Richman 1988), and, as already mentioned, the validity of the local granular rheology of Table 1 is questionable. The use of the Coulomb formula in Eq. (12) leads to a linear profile; hence, the experimental tendency of the free surface to become parallel to the bed in the upwards direction cannot be reproduced, and apart from a region close to the snout, the flow depth is largely overestimated. The Takahashi and the Manning-Strickler formulas strongly overestimate the flow resistance, and therefore the wave height; the opposite for the Newtonian laminar formula. The results obtained with the Voellmy formula are the closest to the experiments, apart from those obtained with Eq. (9). Obviously, we could have improved the agreement between the experiments and the predictions obtained using the above mentioned empirical formulas, by tuning the parameters present in the different expressions (except for the Coulomb formula, whose unrealistic consequences on the wave profile are independent on the choice of $\tan\phi$). The *a priori* choice of the parameters in the formulas, though, highlights the superiority of Eq. (9), that does not require an *ad hoc* parameter adjustment.

Tubino and Lanzoni (1993) reported measurements of the wave height as a function of time, t, for one of their experiments on debris flows of water and 3 mm gravel in a rectangular channel of width, W, equal to 20 cm (67 diameters). For that experiment, they also measured the velocity of the front, that they described as fully saturated, and found it constant and equal to 47.6 cm/s, i.e., $u_m = 2.8$ in dimensionless units; once again, the flow can then be considered steady in a frame of reference moving at constant velocity, with $x = 2.8t$. Unlike the experiments of Iverson et al. (2010), the debris flow propagated over an erodible bed (Fig.1c), whose initial inclination, θ_0, was equal to 17°; this ensures that a no-slip velocity applies at the interface with the bed, but introduces an additional uncertainty in determining the position of the bed itself, represented by b in the sketch of Fig.1c. The local slope of the bed, tanθ, in Eq. (12), can be expressed as

$$\tan\theta = \tan\theta_0 - \frac{\mathrm{d}b}{\mathrm{d}x}, \qquad (13)$$

and an additional equation is required to solve for the evolution of both h and b along x. Eq. (10) provides this additional relation.

Fig.3a,b show the comparisons between the experimental results of Tubino and Lanzoni (1993) and those obtained by numerically solving the system of Eqs.(10), (12) and (13), using again a fourth-order Runge-Kutta method, with Eq. (11) and the five resistance formulas of Table 2 for j, and the boundary conditions $h = b = 0$ at $x = L$. We keep the same values for the parameters in the resistance formulas adopted in the case of Fig.2, but for the parameter a in the Takahashi's formula that, in the case of erodible bed, is supposed to be equal to 0.042 (Takahashi 1991). We take μ_w in Eq. (10) to be equal to 0.39, as suggested by Berzi et al. (2010).

The particle Reynolds number R is about 500 and therefore much greater than $10h^{-1/2}$ for the experiments of Tubino and Lanzoni (1993), given that h is of order ten diameters (Fig.3). The agreement between the experimental and the theoretical wave profile obtained using the theory of Berzi and Jenkins (Fig.3a) is remarkable. Also the shape of the snout is well reproduced in this case, despite the fact that the flow there is thin. This seems to suggest that the local granular rheology of Table 1 holds also in the proximity of the bottom boundary (erodible bed), if the latter acts as a sink of energy to the flow (Jenkins and Askari 1991). The use of Eq. (11) results also in an erodible bed which is substantially unperturbed by the wave propagation (Fig.3a). This is in accordance with the observations of Tubino and Lanzoni (1993), although they did not report direct measurements of the position of the bed. None of the other resistance formulas allows to reproduce the experiments; in particular, the Voellmy formula, that gives good results in the case of the experiments of Iverson et al. (2010), dramatically underestimates the resistances in this case (Fig.3b).

A final test of the theory would consist in evaluating its performance with regards to field data. Rickenmann (1999) compiled data sets of field and laboratory measurements of mean velocity, flow depth and angle of inclination of the bed from different literature sources. Assuming that the data refer to roughly uniform flows, i.e., flows for which the bed slope, tanθ, coincides with the friction slope, *j*, they can be used to assess the validity of the theory. In particular, we make comparisons with Eq. (9) neglecting the term associated with the frictional sidewalls, because the field data refer to natural channels with expected small ratio of flow depth to channel width. The condition of fully saturation is rather exceptional, though; as revealed by the experiments of Armanini et al. (2005), the flow is always over-saturated at mild slopes, i.e., the height of the water is greater than the height of the particles above the bed. Nonetheless, it can be shown that Eq. (9) is representative of the resistances also when the flow is over-saturated, if the concentration *c* is taken to be the bulk value over the entire flow depth (Berzi et al. 2010). Fig.4 shows the comparison between the field and laboratory measurements, reported by Rickenmann (1999), for which R is much greater than $10h^{-1/2}$, and the theoretical predictions of Eq. (9), in terms of the ratio $u_m / h^{3/2}$ against tanθ. The field measurements have been performed on the Torrente Moscardo in Italy (Arattano et al. 1996) and the Jiangia gully in China (Rickenmann, written comm., 2011); the laboratory measurements were performed by Wang and Zhang (1990), Garcia Aragon (1996) and Iverson and LaHusen (1993). The mean diameter of the granular material ranges between 1 mm and 1 cm. Given the usual values of the bulk concentration for debris flows (Takahashi 1991), we take *c* equal to 0.2 and 0.6 in Eq. (9) to draw the two theoretical curves of Fig.4. The values of the other parameters in Eq. (9) are exactly the same used for the comparisons of Fig.2 and 3. Despite all the uncertainties that characterize the measurements, the most of the field and laboratory measurements are in the region between the two curves, and the trend of the $u_m / h^{3/2}$ to increase with the bed slope is notably reproduced by the theory.

Conclusions

This work has focused on the resistance formulas to be used in mathematical models of inertial debris flows, i.e., granular-fluid mixtures for which both the fluid viscous forces and the fluid turbulence does not substantially affect the particle interactions at the micro-mechanical level. For simplicity, we have limited the analysis to fully saturated flows, i.e., flows for which the fluid and particle depths coincide.

The main results of the paper are: (i) the most of real scale debris flows (Iverson 1997) are inertial debris flows, given that the concentration is higher than 40%, so that the fluid turbulence is suppressed, and the particle Reynolds number is much greater than ten times the inverse of the square root of the non-dimensional flow depth, so that the fluid viscous forces are negligible with respect to the particle inertia; (ii) hence, there is no physical justification to adopt, in depth-averaged mathematical models of inertial debris flows, resistance formulas of either ‘viscous’, such as those based on Newtonian, Bingham or Herschel-Bulkley rheologies, or ‘turbulent’ origin, such as the Manning-Strickler or the Voellmy expression; (iii) the particle slip velocity at a rigid bed, i.e., the influence of the bottom boundary, can be ignored only if the flow depth is much greater than ten diameters - this usually applies to real scale events, not to most of the available laboratory experiments on inertial debris flows; (iv) the physically based resistance formulas obtained from the theory of Berzi and Jenkins (2008a,b, 2009) allow to reproduce, in a notable way, both the experimental longitudinal profile of steady waves of water and gravel measured by Iverson et al. (2010) and Tubino and Lanzoni (1993), and the field measurements of real events reported in the literature and collected by Rickenmann (1999); (v) neither the Coulomb (Iverson 1997; Pitman and Le 2005) nor the Takahashi (1991) resistance formula allow to fit the experimental results, raising some doubts about their implementation in mathematical models of debris flows.

Acknowledgements

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Notation

The following symbols are used in the paper:

a = coefficient in the Takahashi's formula [-];

c = particle volume concentration [-];

c_0 = particle volume concentration at a rigid bed [-];

c^* = particle volume concentration at the closest packing [-];

d = particle diameter [m];

D = drag force [-];

g = gravitational acceleration [m/s^2];

h = particle depth over the bed [-];

I = inertial number [-];

j = friction slope [-];

l = mixing length in the fluid turbulent shear stress [-];

L = position of the wave front [-];

n = Manning's coefficient [m$^{1/3}$/s];

p = particle pressure [-];

p_0 = particle pressure at a rigid bed [-];

R = particle Reynolds number [-];

R^* = modified particle Reynolds number [-];

s = particle shear stress [-];

s_0 = particle shear stress at a rigid bed [-];

S = fluid shear stress [-];

St = Stokes number [-];

t = time [-];

T = granular temperature [-];

u = particle velocity [-];

u_0 = particle slip velocity at a rigid bed [-];

u_m = depth-averaged particle velocity [-];

U = fluid velocity [-];

W = channel width [-];

x = coordinate in the flow direction [-];

z = coordinate in the direction perpendicular to the flow [-];

χ = material coefficient [-];

δ = difference between the fluid and the particle velocity in the flow direction [-];

ϕ = Coulomb's basal friction angle [°];

η = fluid viscosity [Pa·s];

$\breve{\mu}$ = yielding value of the particle stress ratio at the bed [-];

μ_w = wall friction coefficient [-];

θ = local angle of inclination of the bed [°];

θ_0 = unperturbed angle of inclination of the erodible bed [°];

ρ = fluid density [kg/m^3];

σ = ratio of particle density over fluid density [-];

ξ = Voellmy's coefficient [m/s^2].

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List of tables

Table 1. Momentum balances and constitutive relations for the steady, uniform, debris flow

Particle momentum balance along x	$s' = -c\sin\theta - D + 2\frac{\mu_w}{W}p$
Particle momentum balance along z	$p' = -c(\sigma-1)\cos\theta/\sigma$
Fluid momentum balance along x	$S' = -(1-c)\sin\theta/\sigma + D$
Drag	$D \equiv \frac{c}{\sigma(1-c)^{3.1}}\left[\frac{3}{10}(\delta^2+3T)^{1/2}+\frac{18.3}{\mathrm{R}}\right]\delta$
Particle rheology	$\frac{s}{p} = \breve{\mu} + \chi I$
Fluid turbulent shear stress	$S = (1-c)l^2U'^2/\sigma$

Table 2. Literature resistance formulas for debris flows

Resistance formula	j
Coulomb	$\tan\phi$
Takahashi	$\frac{25}{4}\frac{0.3a}{\left[(c^*/c)^{1/3}-1\right]^2}\frac{\sigma}{\left[(\sigma-1)c+1\right]}\frac{u_m{}^2}{h^3}$
Newtonian laminar	$\frac{3}{\mathrm{R}^*\left[(\sigma-1)c+1\right]}\frac{u_m}{h^2}$
Manning-Strickler	$\frac{n^2g}{d^{1/3}}\frac{u_m{}^2}{h^{4/3}}$
Voellmy	$\frac{(\sigma-1)c}{(\sigma-1)c+1}\breve{\mu} + \frac{g}{\xi}\frac{u_m{}^2}{h}$

Figure1a

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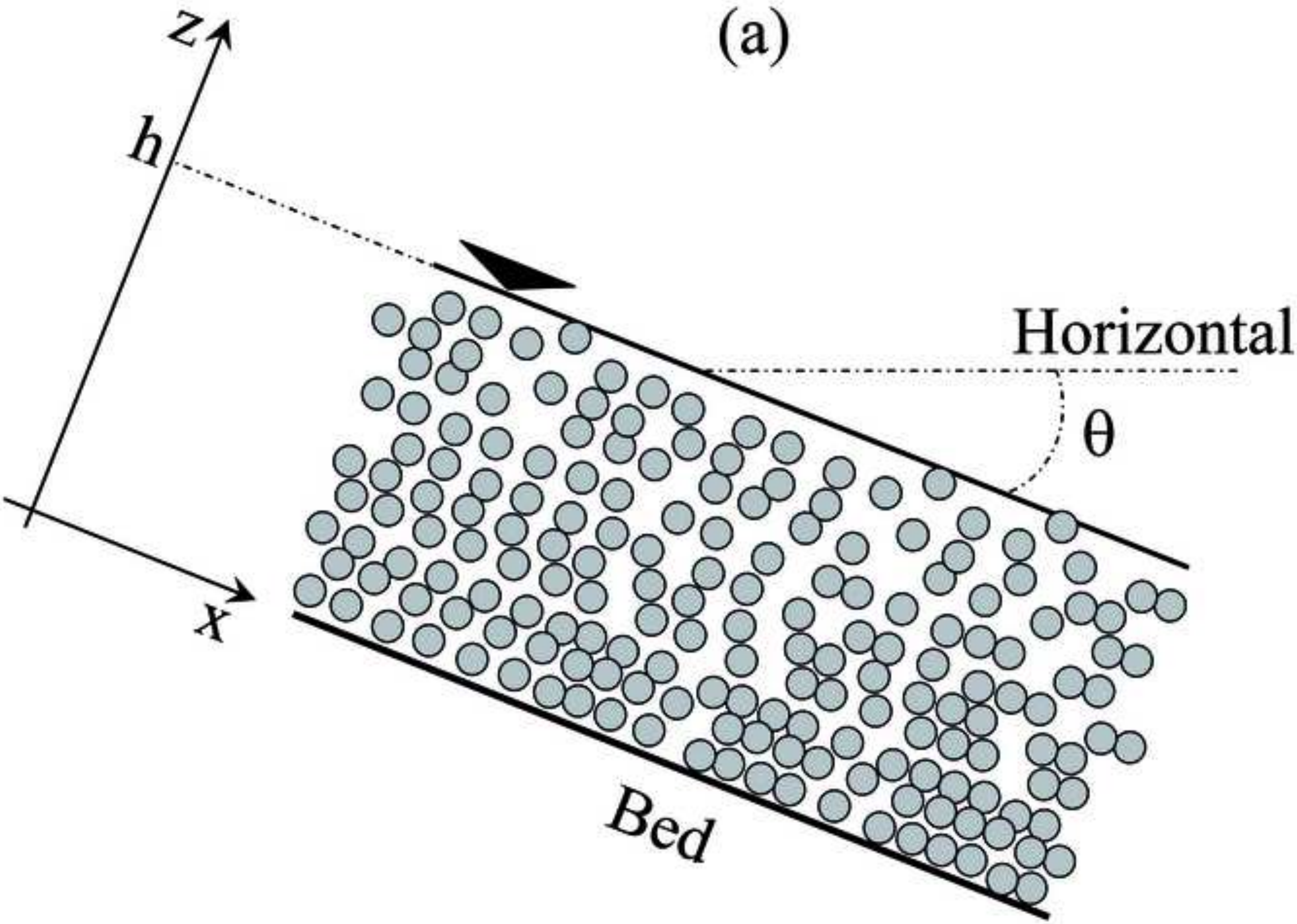

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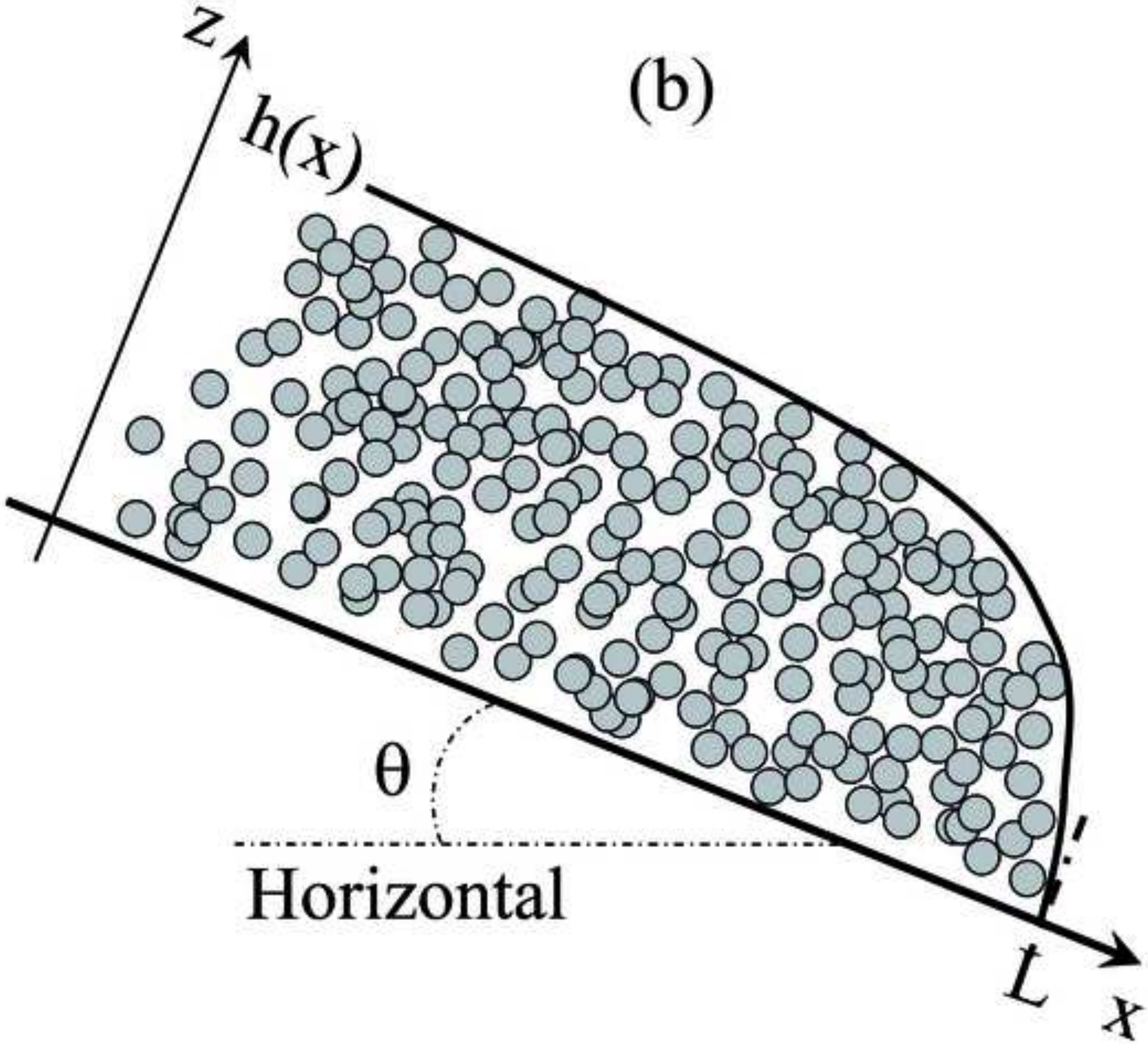

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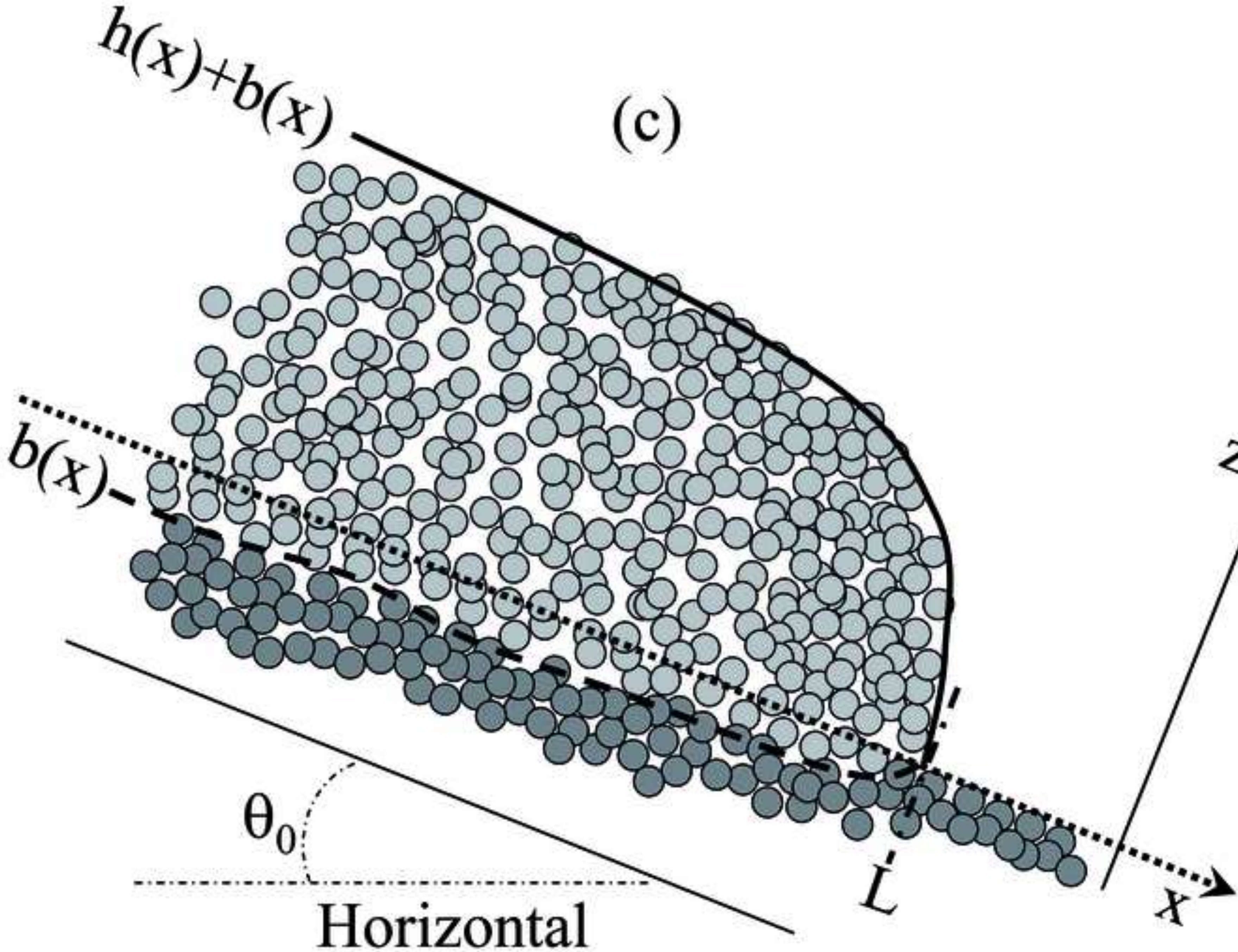

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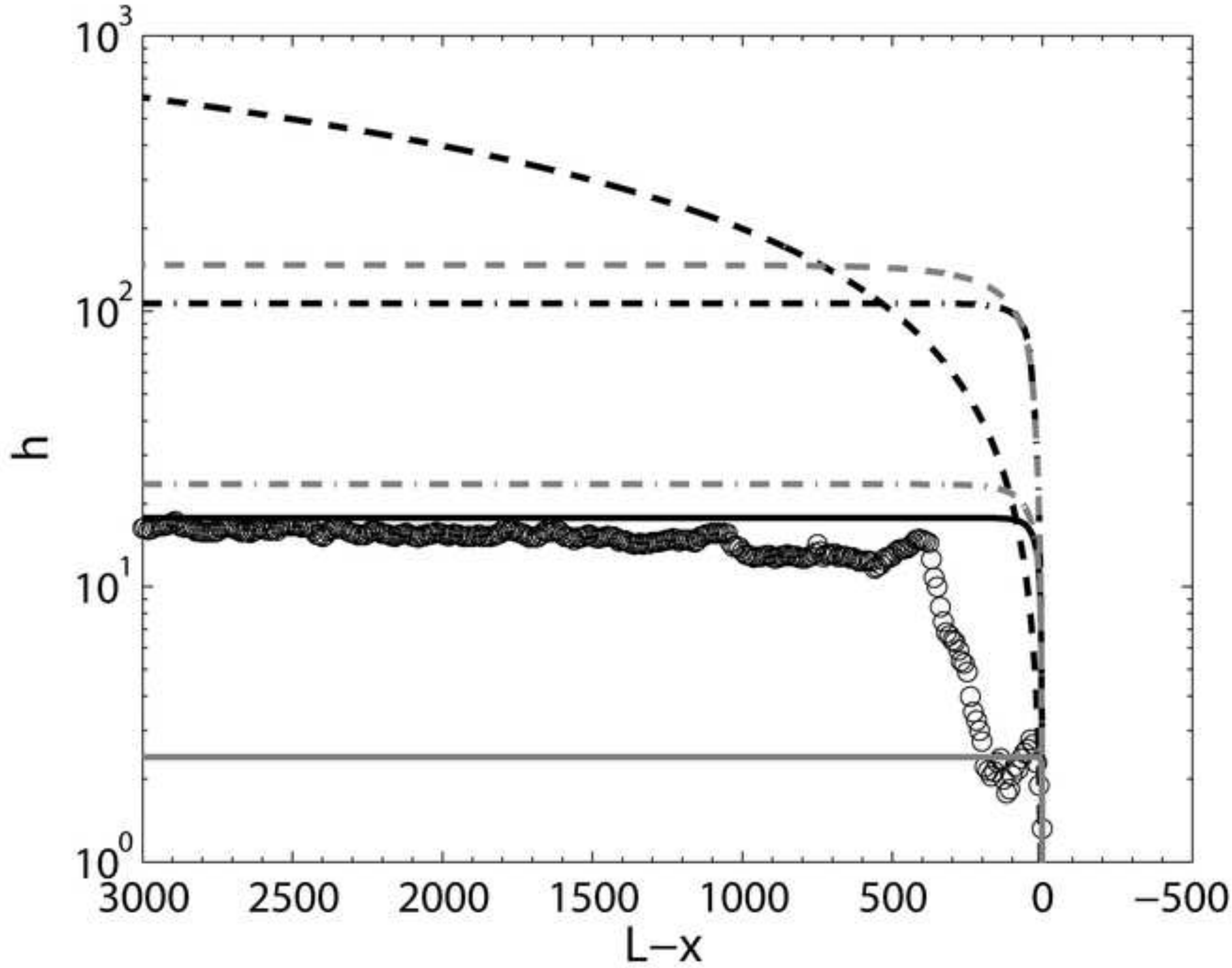

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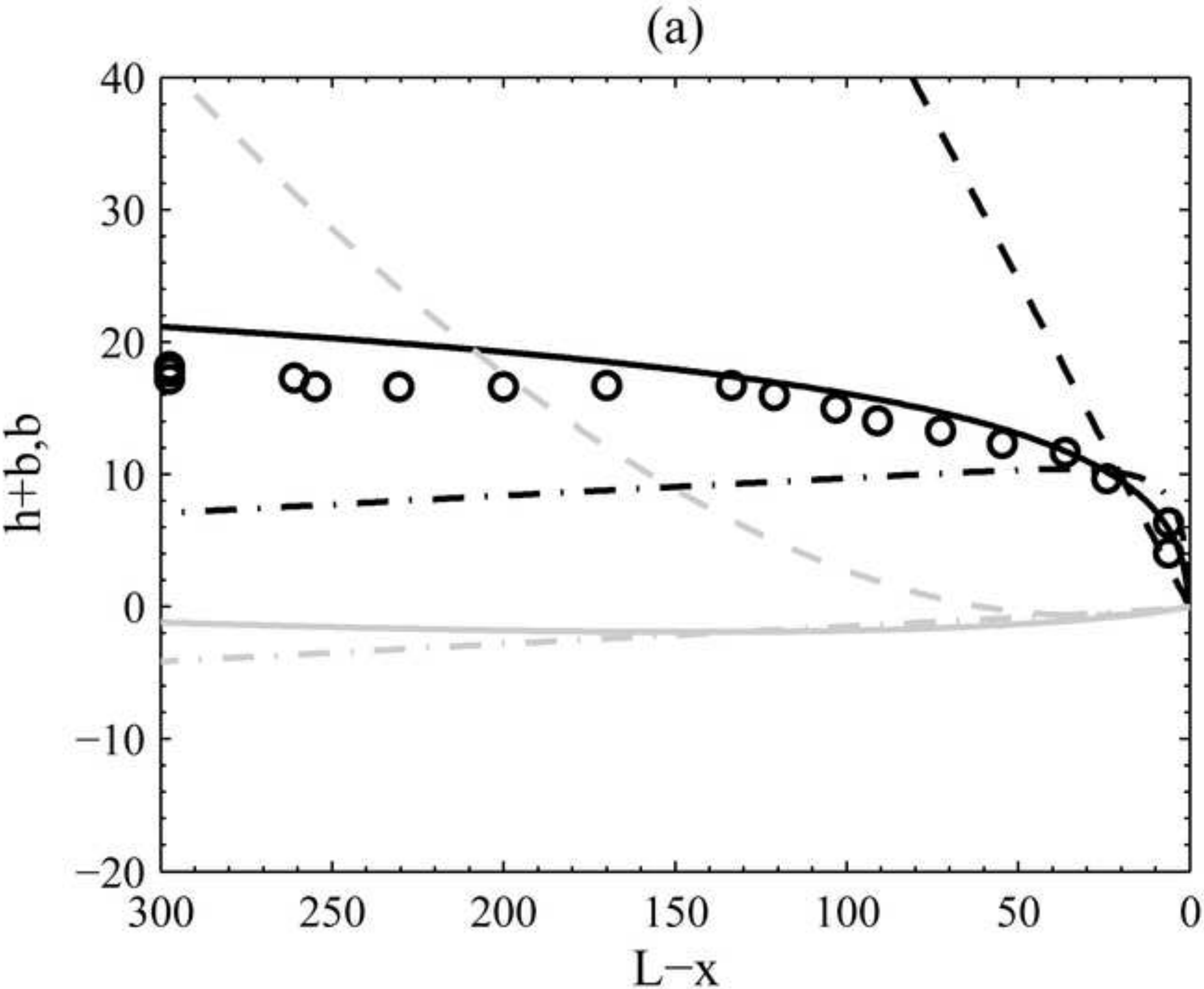

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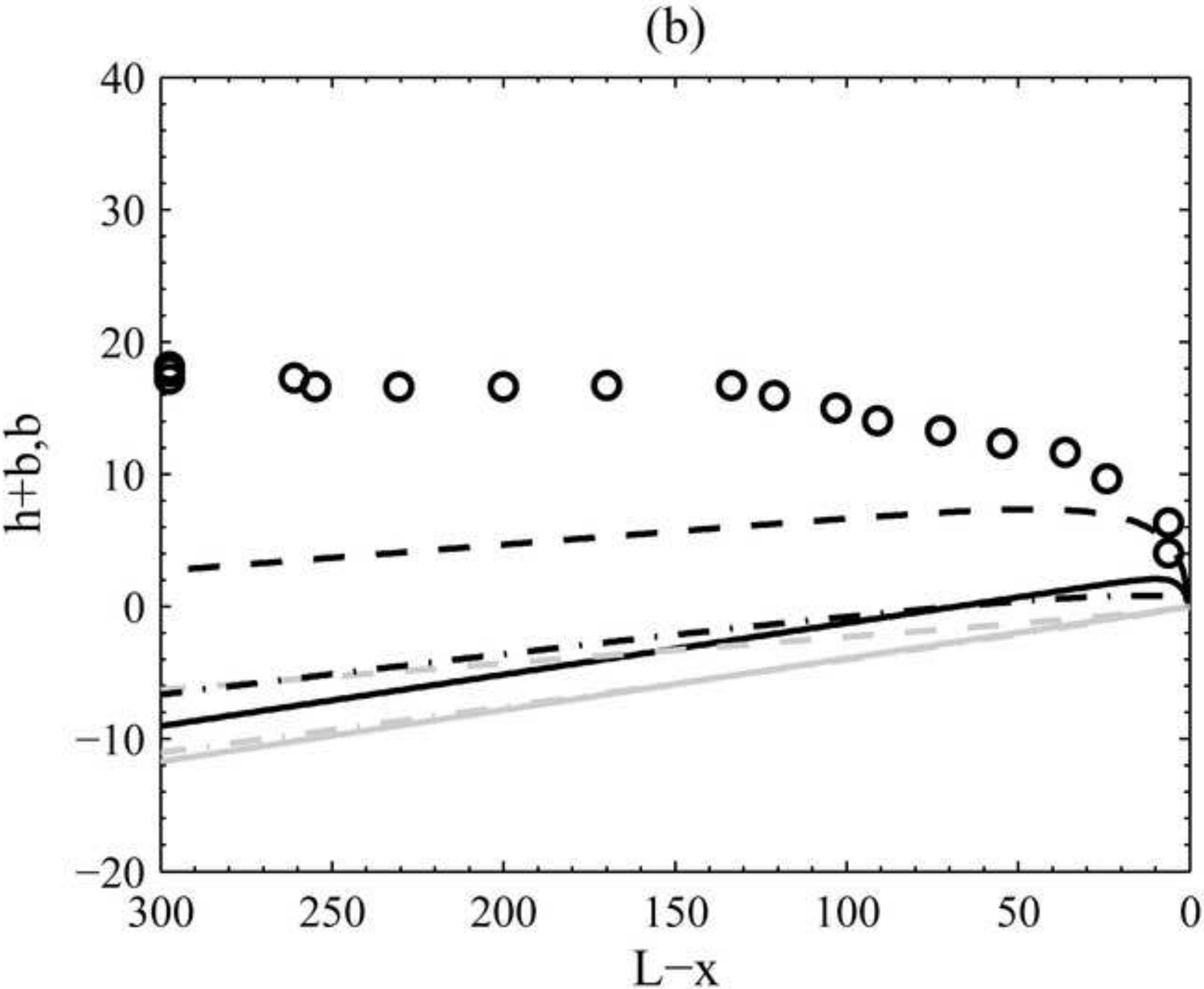

Figure4

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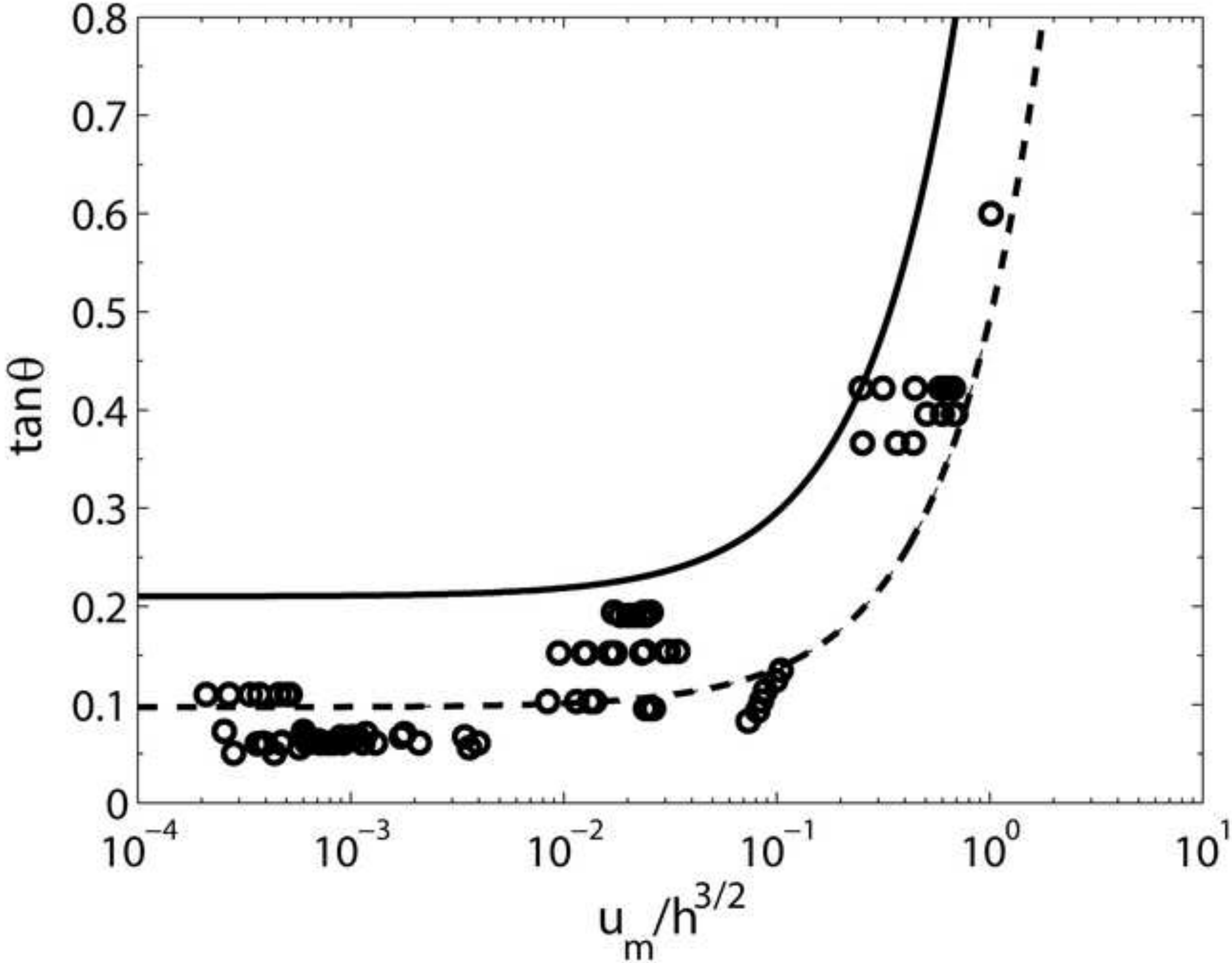

List of figure captions

Figure 1. (a) Steady, uniform, fully saturated debris flow. (b) Steady, non-uniform, fully saturated debris flow over a rigid bed. (c) Steady, non-uniform, fully saturated debris flow over an erodible bed.

Figure 2. Experimental (circles, from Iverson et al. 2010) against theoretical (lines) longitudinal profile of a steady wave over a rigid bed, obtained by solving Eq. (12) with the different expressions for *j*: Eq. (9) (solid black line); Coulomb (dashed black line); Takahashi (dot-dashed black line); Newtonian laminar (solid gray line); Manning-Strickler (dashed gray line); Voellmy (dot-dashed gray line).

Figure 3. (a) Experimental evolution of the free surface (circles, from Tubino and Lanzoni 1993) and theoretical evolution of the free surface (black lines) and the erodible bed (gray lines) for a steady wave over an erodible bed, obtained using: Eq. (11) (solid lines); Coulomb (dashed lines); Takahashi (dot-dashed lines). (b) Same as in Figure 3a, but using: Newtonian laminar (solid lines); Manning-Strickler (dashed lines); Voellmy (dot-dashed lines).

Figure 4. Field and laboratory measurements (circles, see the text for the sources) of the ratio $u_m / h^{3/2}$ against bed slope for inertial debris flows. Also shown are the predictions of Eq. (9), for c = 0.6 (solid line) and c = 0.2 (dashed line).